AN EXTENSION OF THE PARALLELISM OF CLIFFORD

BY

JOHN CLARENCE BELL

B.S., University of Illinois, 1937

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AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1942

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I. GENERAL REMARKS ON R_n

Introduction. The preparation of this paper was begun by viewing an elliptic 3 - space as embedded in a euclidean 4 - space, and then expressing the parallelism of Clifford in the elliptic space as a linear differential parallelism in the coordinates of the embedding space. The resulting differential equations were found to define a teleparallelism in the euclidean space having as its autoparallels the plane equiangular spirals emanating from a single point. Finally a Riemannian space was found having these curves as its geodesics; and in terms of its metric it soon appeared that these curves have many of the properties of the Clifford parallels, and lend themselves to a generalization in spaces of higher dimension.

The results here obtained may be contrasted with those of Whittaker[1], who has obtained a class of three dimensional spaces all admitting a teleparallelism having the same linear differential equations as the parallelism of Clifford. The metric of the Riemannian space here discussed is a special case of the metric employed by Robertson in his articles on world structure[2].

Fundamental and curvature tensors. The purpose of this paper is to present an extension of the parallelism of Clifford to the Riemannian space with the metric

$$(1) \qquad ds^2 = \frac{(dx\ dx)}{(xx)} ,$$

where $(dx\ dx) = \sum_{i=1}^{n} dx^{i2}$, and $(xx) = \sum_{i=1}^{n} x^{i2}$.[3] This space will be denoted by R_n; and only the geometry of its real points will be discussed.

If the x's are interpreted as the rectangular coordi-

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1) E. T. Whittaker: Parallelism and Teleparallelism. London Mathematical Society Journal. v. 5. pp 68 - 80.

2) H. P. Robertson: Kinematics and World Structure. Astrophysical Journal. v. 82. pp 284 - 301.

3) Throughout this paper, if $x^1,\ldots,x^n$ and $y^1,\ldots,y^n$ are any sets of n numbers, then (xy) will be used to denote $\sum_{i=1}^{n} x^i y^i$.

nates of euclidean n - space E_n, a mapping of R_n is obtained which is conformal, since the coefficients of the fundamental form (1) are proportional to those of the euclidean metric. It will frequently be helpful to apply to R_n the terminology of E_n. Thus the hypersurface $(xx) = R^2$ will be called a hypersphere around the origin; the curve $x^i = e^s a^i$ will be called a ray; etc.

The Riemann curvature tensor may be computed to be

$$R_{hijk} = \frac{1}{(xx)^3}[x^i x^j \delta_{hk} + x^k x^h \delta_{ij} - x^i x^k \delta_{hj} - x^h x^j \delta_{ik}] + \frac{1}{(xx)^2}[\delta_{hj}\delta_{ik} - \delta_{hk}\delta_{ij}].$$

For $n > 2$, R_n is not of constant curvature. In fact, if λ and μ are orthogonal unit vectors with components $\lambda^1, \ldots, \lambda^n$ and $\mu^1, \ldots, \mu^n$, the curvature of R_n at x in their orientation is

$$1 - \frac{1}{(xx)^2}[(x\mu)^2 + (x\lambda)^2].$$

The coordinates of x may be interpreted as the components of the unit vector tangent at the point x to the ray from the origin. There is no loss in assuming μ orthogonal to x; i. e., $(x\mu) = 0$. Since $(x\lambda)/(xx)$ is the cosine of the angle between x and λ, it has been shown that

The Riemannian curvature of R_n for the orientation of a given plane at a given point is the square of the sine of the angle between the normal to the hypersphere, with center at the origin, passing through the given point and the normal to the circle of intersection of the given plane with this hypersphere.

This holds for all values of n.

Geodesics. The geodesics of any Riemannian space have the differential equations $\frac{\Delta}{ds}\frac{dx^i}{ds} = 0$, where s is a linear function of the arc length, and the $\frac{\Delta}{ds}\frac{dx^i}{ds}$ are covariant derivatives. In R_n these become, after dropping the superscripts,

$$\frac{d^2x}{ds^2} - 2\frac{\left(x\frac{dx}{ds}\right)}{(xx)}\frac{dx}{ds} + x = 0. \tag{2}$$

By differentiation it can be shown for any solution of these equations, say $x = x(s)$, that $(x\frac{dx}{ds})/(xx)$ is a constant, say k. However, this k may differ for various solutions. It may be verified readily that k is the cosine of the angle the geodesic makes with a ray from the origin; hence for real geodesics, $-1 \leq k \leq +1$. Moreover, if $s' = -s$ is introduced as the

parameter of a geodesic, its differential equations have k replaced by -k; so there is no loss in assuming $0 \leq k \leq 1$. Then any geodesic must be a solution of

$$(2)' \qquad \frac{d^2x}{ds^2} - 2k\frac{dx}{ds} + x = 0,$$

where $0 \leq k \leq 1$; and it must also satisfy $(x\frac{dx}{ds}) = k(xx)$. Taking the general solution of $(2)'$ and requiring it to satisfy this last condition gives the

Theorem 1. The geodesics of R_n are the curves having equations of the form

$$(3) \qquad x = e^{ks}[a\cos(\sqrt{1-k^2}s) + b\sin(\sqrt{1-k^2}s)],$$

where $0 \leq k \leq 1$, and a and b are n x 1 constant matrices such that $(aa) = (bb)$ and $(ab) = 0$. In this form, the parameter s represents arc length along the geodesic. In the euclidean map these are curves lying in planes through the origin, and are either circles with center at the origin (k=0), or equiangular spirals with center at the origin (0<k<1), or rays from the origin (k=1).

Noting that the metric of R_n imposes the same metric on hyperspheres around the origin as the metric of E_n imposes on the unit hypersphere around the origin, it is seen that in R_n the hyperspheres around the origin are totally geodesic hypersurfaces having the constant curvature +1.

Geodesic distances between any two points. Inasmuch as the geodesics of R_n are plane curves, the distances along them between two arbitrary points y and z may be determined by considering the geometry in the plane through y, z, and the origin. Letting $\rho > 0$ and θ be polar coordinates in this plane, the metric of R_n is found to induce in the plane the metric

$$ds^2 = (d\log\rho)^2 + (d\theta)^2.$$

To obtain a single valued geodesic distance between y and z, the plane must be regarded as consisting of infinitely many layers, each layer corresponding to the range $2\pi m \leq \theta \leq 2\pi(m+1)$, m being any integer. Then corresponding to definite determinations (ρ_1,θ_1), (ρ_2,θ_2) of the polar coordinates of y and z there is one geodesic joining them. Since $\log\rho$ and θ are "cartesian" coordinates, the distance between the points, say d, measured along that geodesic is given by

$$d^2 = (\theta_2-\theta_1)^2 + (\log\frac{\rho_2}{\rho_1})^2.$$

In terms of the coordinates of R_n this gives the

Theorem 2. The geodesic distance between any two points y and z is an infinitely many valued function given by

$$(4) \qquad d = \left\{ \left[2m\pi + \arccos \frac{(yz)}{\sqrt{(yy)(zz)}} \right]^2 + \left[\log \sqrt{\frac{(zz)}{(yy)}} \right]^2 \right\}^{\frac{1}{2}},$$

where the principal value is used for the inverse cosine, and m is an integer.

Corresponding to a definite choice of the geodesic distance between y and z there is, in general, a unique geodesic through them whose arc, between them, is the given determination of the distance. In fact, if $(yy) \neq (zz)$ (this being a matter of notation), and if $z \neq \rho y$ where ρ is scalar, the geodesic is unique and its equations are given by (3) where

$$a = y, \quad b = \frac{(yy)z - (yz)y}{\sqrt{(yy)(zz) - (yz)^2}}, \quad k = \frac{1}{d_1} \log \sqrt{\frac{(zz)}{(yy)}},$$

and d_1 is the given determination of the distance. If $z = \rho y$, the geodesic is not unique except if $\rho > 1$ and the distance be that value corresponding to m = 0, in which case the geodesic is the ray $x = e^s y$.

The value of the distance corresponding to m = 0 will be called the principal geodesic distance; and any corresponding geodesic will be called a principal geodesic.

Motions. If a point transformation in R_n, $x' = x'(x)$, is a motion, it must preserve the distance element. Then

$$(dx'\ dx') = \frac{(x'x')}{(xx)} (dx\ dx),$$

so that in E_n it must represent a conformal transformation. Bianchi [1] has shown, in extending the conformal transformation theorem of Liouville, that any conformal transformation in E_n can only be of one of two types:

$$(a) \qquad x' = M(x+a), \text{ or}$$

$$(b) \qquad x' = \frac{M(x+a)}{(\overline{x+a}\ x+a)} + b,$$

where a and b are constant n x 1 matrices, and M is an n x n constant matrix which is orthogonal to within a constant factor. Such a matrix M will be called almost orthogonal. In order that the transformations (a) and (b) be motions in

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[1] L. Bianchi: Lezioni di Geometria Differenziale, (1923 ed.), vol. 2, part 2, p. 469. A conformal transformation C on E_n $(n > 2)$ may be a motion of E_n. If it is not, then it is such that CTR = M, where T is a translation, R an inversion in a hypersphere with center at the origin, and M a motion. Then $C = MRT^{-1}$.

R_n, it is found to be necessary and sufficient that in (a) $a = 0$, while in (b) $a = b = 0$. Finally if the degrees of freedom in choosing the matrices M be counted, one has the

Theorem 3. The space R_n, with $n > 2$, admits a group of motions consisting of two $\frac{n^2 - n + 2}{2}$ - parameter sets. The motions are of the two types

$$(5)\quad \begin{matrix} (a) & x' = Mx \\ (b) & x' = Mx/(xx) \end{matrix},$$

where M is any matrix almost orthogonal. The motions of the former type form a subgroup.

Since motions of the type (5)(a) do not change the positive sense of the parameter s on a non-circular geodesic, while those of type (5)(b) do, the former will be called direct motions and the latter indirect.

II. C - CONGRUENCES IN R_n

The class of C-congruences. The congruences of Clifford parallels in elliptic 3 - space are curvewise invariant (i.e., each curve is invariant) under a one-parameter group of displacements. In order to extend this property to a class of congruences in R_n, one may begin by stating the

Definition. A displacement in R_n is a motion for which some determination of the geodesic distance between a point and its transform is the same for all points.

From equation (4) there follows the

Theorem 4. A direct motion in R_n is a displacement if and only if its matrix is skew (i.e., the sum of a skew-symmetric and a scalar matrix). No indirect motion can be a displacement.

Moreover, in order to exhibit a direct motion with a skew matrix as a displacement, the same determination of the geodesic distance must be used between all the paired points. It will be shown later that there are enough skew matrices to make non-vacuous the

Definition. A congruence of geodesics in R_n is a C-congruence if there exists a continuous one-parameter family of displacements which leave the congruence curvewise invariant. (It is assumed further that this family has at least two distinct displacements.)

It can be shown first that all the geodesics of a C-con-

gruence make the same angle with the rays from the origin. For if a displacement leaves invariant one circular geodesic then all the geodesics it leaves invariant are circular. Hence if k vanishes along one curve of a C-congruence, it must vanish along all. Then consider a C-congruence Γ having no curves with $k = 0$. A displacement $x' = Mx$, where $M^TM = Ih^2$ and $h>0$, leaving Γ curvewise invariant transforms one of its geodesics, say (3), into

(3)' $\qquad x' = e^{kt}[Ma\cos(\sqrt{1-k^2}\,t) + Mb\sin(\sqrt{1-k^2}\,t)]$,

which must be the same as (3) under some correspondence $s=s(t)$. Using the condition $(xx) = (x'x')$, it is found that $s = t+s_o$, where $s_o = \frac{1}{k}\log h$. Multiplying $x(s_o) = x'(0)$ by a^T gives

$$\cos\left(\frac{\sqrt{1-k^2}}{k}\log h\right) = \frac{q}{h},$$

where q is the diagonal element of M. If there were in Γ two geodesics with differing k's, say k_1 and k_2, then

$$\left(\frac{\sqrt{1-k_1^2}}{k_1} \mp \frac{\sqrt{1-k_2^2}}{k_2}\right)\log h = 2n\pi,$$

where n is an integer. Then h could assume only a discrete set of values. However, h is a continuous function of a parameter of the given family of displacements, and therefore could take only one value. This would lead to a contradiction of the existence of more than one displacement in the family. Hence k is constant throughout .

Second, it can be shown that if a displacement with a non-scalar matrix leaves a C-congruence curvewise invariant, then if the geodesics are expressed in the form (3) each has $b = \pm(M-Iq)a/p$, where q is the diagonal element of $M, M^TM=Ih^2$, $h>0$, $p=\sqrt{h^2-q^2}$, and the ambiguous sign is the same throughout the congruence. Regardless of the value of k, the transform of any geodesic (3) is (3)', and the correspondence must be of the form $s = \mp t + s_o$. It follows from $x(s_o) = x'(0)$ that

(6) $\qquad h[a\cos(\sqrt{1-k^2}s_o) + b\sin(\sqrt{1-k^2}s_o)] = Ma.$

Multiplying by a^T gives as before that $\cos(\sqrt{1-k^2}s_o) = q/h$. Then $\sin(\sqrt{1-k^2}s_o) = \pm p/h$, and $b = \pm(M-Iq)a/p$. If $k\neq 0$, then $\sqrt{1-k^2}s_o = \frac{1}{k}\sqrt{1-k^2}\log h$, so that the sign is the same throughout the congruence. If $k=0$, it is merely a matter of choosing a positive sense on each geodesic to insure the same result. It is trivial that if every displacement leaving a C-congruence curvewise invariant has a scalar matrix, the congruence consists of rays from the origin.

If $b = -(M-Iq)a/p$, then one can still write it in the

form $b = +(M-Iq)a/p$ by replacing M by its transpose. Then any C-congruence must have the form

$$x = e^{ks}[a\cos(\sqrt{1-k^2}s) + \frac{M-Iq}{p}a\sin(\sqrt{1-k^2}s)], \tag{7}$$

where k and the non-scalar skew matrix M are constant in the entire congruence, $M^TM = Ih^2$, q is the diagonal element of M, and $p = \sqrt{h^2-q^2}$; or else the congruence must have the form

$$x = e^{s}a. \tag{8}$$

Finally, a congruence of the form (7) is curvewise invariant under displacements of the one-parameter continuous family

$$x' = M'x, \text{ where}$$

$$M' = e^{k\sigma}[I\cos(\sqrt{1-k^2}\sigma) + \frac{M-Iq}{p}\sin(\sqrt{1-k^2}\sigma)], \tag{9}$$

while the congruence (8) is curvewise invariant under

$$x' = e^{\sigma}x. \tag{10}$$

These results give the theorem

Theorem 5. A congruence of geodesics is a C-congruence if and only if each has the form (7) or each has the form (8).

It may further be remarked that in both (9) and (10) the parameter σ represents the distance between any point and its transform if this distance is measured along a geodesic of the associated invariant congruence.

Displacements associated with a C-congruence. A complement to the preceding theorem is

Theorem 6. A necessary and sufficient condition that a displacement leave curvewise invariant a C-congruence

(i) of form (7) is that the displacement have the form (9);
(ii) of form (8) is that the displacement have the form (10).

Only the necessity condition of (i) requires further proof. If any displacement $x' = M'x$ be assumed to leave invariant the geodesic of (7) which passes through the arbitrary point a, it is found by a process similar to that used in obtaining (7) that

$$M'a = e^{ks_o}[a\cos(\sqrt{1-k^2}s_o) + \frac{M-Iq}{p}a\sin(\sqrt{1-k^2}s_o)]. \tag{11}$$

If $k \neq 0$, then $s_o = \frac{1}{k}\log h'$, which is independent of a. If $k=0$, then $s_o = 2m\pi \pm \arccos\frac{q'}{h'}$, so that $\cos s_o$ can not vary with a, and $\sin s_o$ can assume only discrete values. But then also $\sin s_o$ is continuous in a, and so must be independent of a. Therefore, (11) is an identity in a, and M' has the form (9) with $\sigma = s_o$.

Corollary 1. The displacements leaving a C-congruence curvewise invariant form a continuous one-parameter group.

Corollary 2. If a given geodesic belongs to a C-congruence, there exists exactly one displacement leaving the congruence curvewise invariant and carrying any point of the given geodesic into any other one of its points.

Finally, it can be shown tnat for any given displacement, except one of the form $x' = -qx$, $q>0$, in R_n of odd dimension, it is possible to determine a C-congruence which that displacement leaves curvewise invariant. The proof of the case $x' = -qx$ in R_n of even dimension depends upon the anticipated result that in any even dimension there exist non-scalar skew matrices which are almost orthogonal. Thus the displacements in R_n of even dimension are bonafide analogues of translations in euclidean space.

Equidistant geodesics in C-congruences. Any two curves of the same C-congruence will now be shown to be so related that they may be described as being equidistant. Suppose the congruence is (7) with $k \neq 1$. Let its curves C_1 through the point a and C_2 through b have parameters s and t respectively. If $f = f(t,a,b)$ is four times the square of the distance[1] from a to a variable point of C_2, then

$$(12)\quad f(t,a,b) = 4\left[\arccos\frac{(ab)\cos(\sqrt{1-k^2}t) + a'M - Iq}{\sqrt{(aa)(bb)}}\right]^2 + \left[2kt + \log\frac{(bb)}{(aa)}\right]^2.$$

Now $f(t,a,b) = f(-t,b,a)$, and $f(s,b,a)$ is four times the square of the distance from b to a variable point of C_1. Hence if the distance from a to a variable point of C_2 is minimum for b, then the distance from b to a variable point of C_1 is minimum for a. If either distance is an absolute minimum, then both are. Two points so related will be said to be at mutual minimum (or absolute minimum) distance. Moreover, it is readily shown that f must assume an absolute minimum value as t varies; i.e., there must be a point on C_2 at an absolute minimum distance from a. Results similar to these are readily seen to hold for curves of a C-congruence with $k = 1$.

Now let a on C_1 and b on C_2 be at mutual minimum distance, while C_1 and C_2 belong to any one C-congruence. Establish a 1 - 1 point correspondence between C_1 and C_2 by

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1) Distances are principal geodesic distances unless otherwise stated.

making correspond to a generic point a' of C_1 the point b' of C_2 into which b is carried by the displacement which carries a into a'. This correspondence yields at once the

Theorem 7. For any two geodesics of the same C-congruence there exists a 1-1 correspondence between their points such that all pairs of corresponding points are at the same mutual minimum distance. For at least one such correspondence this distance is an absolute minimum. Moreover, if a given pair of points are at mutual minimum distance, there exists a correspondence in which they are corresponding points.

This theorem describes one sense in which C_1 and C_2 are equidistant. Another description may be had by determining an analytical condition for the points a and b to be at mutual minimum distance. A necessary condition for this if $k \neq 1$ is that $f_t(0)=0$, unless the function f assumes a minimum value at a point where its derivative is discontinuous or undefined. If $k = 1$, then a necessary condition is that $(aa) = (bb)$.

Return to the case $k \neq 1$. If z is a variable point on C_2, the derivative f_t is continuous and defined at z unless $a = \rho z$, where ρ is scalar. If C_1 and C_2 are not coplanar, then from (7) it can be shown that no points of C_2 lie in the plane of C_1; i.e., for no z can $a = \rho z$. If C_1 and C_2 are coplanar, and $k \neq 0$, then it can be shown that although f_t has discontinuities, none of them can occur at points where f has a minimum value. The proof of this can be facilitated by referring C_2 to an initial point b_1, chosen as the intersection of the ray $\overline{0\ a}$ with the curve C_2 and which, moreover, is between 0 and a at the smallest possible distance from a. With this choice of initial point for C_2, the discontinuities of f_t occur at the points $\sqrt{1-k^2}t = n\pi$, where n is an integer. Evaluation of left-hand and right-hand derivatives of f at these points shows that none of them correspond to minimum values of f. Thus if C_1 and C_2 are neither rays nor coplanar circles, their points a and b are at mutual minimum distance only if $f_t(0) = 0$. This is equivalent to

$$(13)\quad 2\sqrt{1-k^2}\left[\arccos\frac{(ab)}{\sqrt{(aa)(bb)}}\right]\frac{a^T\frac{M-Iq}{p}b}{\sqrt{(aa)(bb)-(ab)^2}}-k\log\frac{(bb)}{(aa)}=0.$$

Finally, if C_1 and C_2 are coplanar circles, their points a and b are at mutual minimum distance only if $a = \rho b$, where ρ is a scalar > 0. Having found in all cases, analytical conditions for a and b to be at mutual minimum distance, one

may by taking the principal geodesic from a to b, by determining the angles it makes with C_1 and C_2, and by using the properties of displacements verify the

Theorem 8. For any two given curves of a given C-congruence, a principal geodesic joining any two points which are at mutual minimum distance is orthogonal to both the given curves. The displacements leaving the congruence curvewise invariant carry this orthogonal geodesic into all the geodesics of a continuous one-parameter family of geodesics which are orthogonal to both the given curves and have a common arc length between their intersections with them.

This theorem describes a second sense in which C_1 and C_2 are equidistant. In addition it may be remarked that C_1 and C_2 and any two geodesics of an orthogonal family form a configuration which may properly be called a rectangle. Its angles are right angles, and its opposite sides are of equal length.

C-fields of vectors. In order to speak of vector fields rather than of curves, one may use the

Definition. A C-field of vectors in R_n is a field of vectors all of the same magnitude and all having unit components tangent in the same sense to curves of the same C-congruence through their respective points of definition. (Recall that for all C-congruences there is a positive sense of the parameter on each curve.)

Theorem 9. The vectors of a field comprise a C-field if and only if each member λ of the field satisfies an equation of the form $\lambda = Mx$, where M is skew, almost orthogonal, and constant throughout the field, and x is the point of definition of λ.

That every C-field has this form is a consequence of the definition and equations (7) and (8). Conversely, if every vector of a field satisfies an equation $\lambda = Mx$ as required by the theorem, then all its vectors have the same magnitude, and if $M \neq Iq$ they are tangent to the curves of the C-congruence

$$x = e^{qs/h}\left[a\cos\left(\sqrt{1-(q/h)^2}\,s\right) + \frac{M-Iq}{p}a\sin\left(\sqrt{1-(q/h)^2}\,s\right)\right],$$

where h, q, p are the familiar quantities associated with M. (This can be called the integral C-congruence of the field.) If M=Iq, then all the vectors are tangent to congruence (8).

Skew, orthogonal matrices. All C-congruences, C-fields,

and displacements are effectively determined by the

Theorem 10. The real, skew matrices, almost orthogonal, which are of even order are the matrices of the form $Iq + pUCU^{-1}$, where p and q are real scalars, U is a real orthogonal matrix, and

$$(14) \qquad C = \begin{pmatrix} 0 & e_1 & 0 & 0 & . & . & . \\ -e_1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & e_2 & . & . & . \\ 0 & 0 & -e_2 & 0 & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & 0 & e_{\frac{n}{2}} \\ . & . & . & . & . & -e_{\frac{n}{2}} & 0 \end{pmatrix}, e_\alpha = \pm 1, \alpha = 1, \ldots, \tfrac{n}{2} .$$

Those of odd order are the real scalar matrices.

Evidently, all matrices of the form $Iq + pUCU^{-1}$ are real, skew, and almost orthogonal. The converse may be shown from the known fact that any real skew-symmetric matrix S has the form UC_oU^{-1} where U is a real orthogonal matrix and C_o has the form of C but with the e_α's replaced by any real constants c_α, and a row and column of zeros added if n is odd[1]. Using such an S to form $M = Iq+S$, and requiring M to be almost orthogonal shows that the c_α's are equal if n is even, and $C_o = 0$ if n is odd.

If $n = 4$, M must be of one of the forms

$$(15) \qquad M_L = \begin{pmatrix} A & -B & -C & -D \\ B & A & D & -C \\ C & -D & A & B \\ D & C & -B & A \end{pmatrix}, \text{ or } M_R = \begin{pmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{pmatrix} .$$

These matrices can be made the basis for left-hand (M_L) and right-hand (M_R) parallelisms of Clifford in elliptic 3-space[2].

Commutative displacements. Three properties of C-congruences arising from commutativity of matrices which are readily shown from the standard forms for the congruences are:

(i) If the matrices of two displacements, neither being a dilation, are commutative, then each displacement permutes the curves of the invariant C-congruences of the other.

(ii) If either of two displacements permutes a proper (i.e., not consisting of circles or rays) invariant C-con-

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1) Cf. J.H.M. Wedderburn, Lectures on Matrices, Am. Math. Soc. Coll. Pub., vol. 17, p. 91.

2) L. Bianchi, op. cit., pp. 517, 521, ff.

gruence of the other, the displacements are commutative.

(iii) If a displacement, not a dilation, with an invariant C-congruence Γ_1 permutes the curves of a proper C-congruence Γ_2, then every displacement leaving one of these congruences curvewise invariant permutes the curves of the other.

Invariant congruences of general motions. Although the C-congruences were defined in terms of a restricted type of motions, displacements, they occupy an essentially unique position as invariant congruences of families of motions.

Theorem 11. The only direct motions in R_n which leave curvewise invariant a congruence of geodesics containing no circles are the displacements.

For any direct motion this can be proved by considering its effect on any geodesic (3). Similarly to (6), it is found

$$h[\, a\cos(\sqrt{1-k^2}s_0) + b\sin(\sqrt{1-k^2}s_0)\,] = Ma\ ;$$
$$h[-a\sin(\sqrt{1-k^2}s_0) + b\cos(\sqrt{1-k^2}s_0)\,] = Mb\ .$$

These equations may be combined to yield

$$[(M^T+M)(aa) - 2I(a\ Ma)]a = 0\ ,$$

which must be an identity in a. It follows that M is skew.

The above theorem holds even if the congruence contains circles, provided the motion does not reverse sense on any of them. Finally, purely geometrical reasons can be given for

Theorem 12. The only congruence of geodesics in R_n of which each geodesic is invariant under two distinct indirect motions is the congruence of rays from the origin.

III. PARALLELISMS IN R_n

C-parallelisms in R_4. Every real skew matrix of order 4 which is almost orthogonal must be of one of the forms M_L or M_R as in (15). These matrices have the property that a matrix equation $\lambda = Mx$, where λ and x are any 4 x 1 matrices except that $x \neq 0$, completely determines the coefficients of M. In fact, $\lambda = M_R x$ implies

$$(xx)\ A = (x\lambda)$$
$$(xx)\ B = (x^1\lambda^2 - x^2\lambda^1) + (x^3\lambda^4 - x^4\lambda^3)$$
$$(xx)\ C = (x^1\lambda^3 - x^3\lambda^1) + (x^4\lambda^2 - x^2\lambda^4)$$
$$(xx)\ D = (x^1\lambda^4 - x^4\lambda^1) + (x^2\lambda^3 - x^3\lambda^2);$$

and $\lambda = M_L x$ implies these same equations but for changes in signs. Thus in R_4, any given vector at any given point deter-

mines exactly two C-fields, which may be described as left-hand or right-hand parallel C-fields according as M_L or M_R is used in determining them. The totality of fields of either type may be said to define a C-parallelism in R_4.

Solving for the constants A, B, C, D in the above equations, differentiating, and solving for $d\lambda$ yields in any case equations of the form $d\lambda^i + \Gamma^i_{jk}\, dx^j \lambda^k = 0$, where the coefficients Γ^i_{jk} are rational functions of x, skew-symmetric in j and k for $j \neq k$. Hence a C-parallelism in R_4 determines in that space an asymmetric linear connection.

To obtain a geommetrical description of this parallelism, note first that all vectors of a parallel field which are defined at points of the same ray from the origin are parallel in the euclidean sense in the euclidean map, but magnitudes vary directly with the euclidean distance from the origin. Second, a vector $\lambda = Mx$ where $M = M(ABCD)$ has as its component in the hypersphere around the origin passing through x the vector $\mu = Nx$, where $N = M(QBCD)$. If x varies over the points of the hypersphere, the vectors $\mu = Nx$ all lie in the hypersphere (an elliptic 3 - space) and comprise a field of vectors parallel in the classic sense of Clifford. Such a field has known geometrical descriptions[1], so the description of the parallelisms in R_4 is complete.

Since every matrix M_L is commutative with every matrix M_R (as may be verified), any displacement in R_4 which leaves a left-hand (or right-hand) C-congruence curvewise invariant permutes the geodesics of any right-hand (or left-hand) C-congruence. Thus, starting with any pair of orthogonal geodesics g_1 and h_1, carrying g_1 into g_2 by a left-hand displacement leaving h_1 invariant, and carrying h_1 into h_2 by a right-hand displacement leaving g_1 invariant, one can build configurations describable as rectangular parallelograms.

C-parallelisms in R_n. Since in spaces of dimension higher than four the matrix equation $\lambda = Mx$ need not have only a finite number of solutions M one is lead to state the

Definition. If in R_n there exists a class of C-fields such that any vector at any point belongs to exactly one of these fields, then these will be called C-parallel fields, and the totality of them a C-parallelism in R_n.

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1) Cf. L. Bianchi, op. cit., p. 521.

An immediate consequence of this and Theorem 9 is

Theorem 13. A space R_n admits a C-parallelism if and only if there exists a class of n-rowed, skew matrices, almost orthogonal, such that any vector λ at any point x satisfies exactly one equation $\lambda = Mx$, M being a matrix of the class.

C-parallelisms in R_8. At least sixteen classes of matrices can be used to define C-parallelisms in R_8. One class is

$$M_1 = \begin{pmatrix} A & -B & -C & -D & -E & -F & -G & -H \\ B & A & -D & C & F & -E & H & -G \\ C & D & A & -B & G & -H & -E & F \\ D & -C & B & A & H & G & -F & -E \\ E & -F & -G & -H & A & B & C & D \\ F & E & H & -G & -B & A & D & -C \\ G & -H & E & F & -C & -D & A & B \\ H & G & -F & E & -D & C & -B & A \end{pmatrix},$$

where A, B,...,H represent independent parameters. If [i,j] represents $(x^i\lambda^j - x^j\lambda^i)$, solving $\lambda = M_1 x$ for A,B,...,H yields

(xx)A=(xλ) (xx)E=[1,5]+[2,6]+[3,7]+[4,8],
(xx)B=[1,2]+[3,4]-[5,6]-[7,8] (xx)F=[1,6]-[2,5]-[3,8]+[4,7],
(xx)C=[1,3]-[2,4]-[5,7]+[6,8] (xx)G=[1,7]+[2,8]-[3,5]-[4,6],
(xx)D=[1,4]+[2,3]-[5,8]-[6,7] (xx)H=[1,8]-[2,7]+[3,6]-[4,5].

To obtain a more compact representation, let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and let (ijk) represent the cross product $P_i x P_j x P_k$. Then
$M_1 = A(444)+B(243)+C(323)+D(123)+E(442)+F(231)+G(421)+H(211)$.
Multiplying $\lambda = M_1 x$ on the left by $x^T(243)$ gives $(xx)B = -x^T(243)\lambda$.
Similarly $(xx)C = -x^T(323)\lambda$, $(xx)D = -x^T(123)\lambda$, $(xx)E = -x^T(442)\lambda$,
$(xx)F = -x^T(231)\lambda$, $(xx)G = -x^T(421)\lambda$, $(xx)H = -x^T(211)\lambda$.
This, with $(xx)A = (x\lambda)$, is equivalent to the previous solution.

Other acceptable classes in similar representation are
$M_2 = A(444)+B(243)+C(324)+D(124)+E(432)+F(241)+G(412)+H(222)$,
$M_3 = A(444)+B(244)+C(323)+D(124)+E(342)+F(132)+G(321)+H(112)$,
$M_4 = A(444)+B(244)+C(324)+D(123)+E(332)+F(142)+G(312)+H(121)$,
$M_5 = A(444)+B(233)+C(423)+D(213)+E(442)+F(241)+G(321)+H(121)$,
$M_6 = A(444)+B(233)+C(424)+D(214)+E(432)+F(231)+G(312)+H(112)$,
$M_7 = A(444)+B(234)+C(424)+D(213)+E(332)+F(132)+G(412)+H(211)$,
$M_8 = A(444)+B(234)+C(423)+D(214)+E(342)+F(142)+G(421)+H(222)$,
and also $M_{i+8} = \frac{1}{4}\sum_{j=1}^{8} M_j - M_i$, $i = 1,\ldots,8$.

All the matrices M_i, $i = 1,\ldots,16$ are the same as M_1 but

for signs of the various elements. All the equations $\lambda = M_i x$ are readily solved for the parameters A,...,H; and the solutions may be represented by bracket expressions as for $\lambda = M_1 x$ but with signs of the brackets varying.

A certain connection in R_n. In any R_n of even dimension a connection can be defined which is closely related to C-parallelisms. This is $\lambda = Dx$, where $D = qI + pBCB$, p and q being real scalars, B is a variable matrix $(2b_i b_j - \delta_{ij})$[1] where the b's are real numbers satisfying (bb)=1, and C is any matrix of the form (14).

The matrices D are real, skew, and almost orthogonal. If c_{ij} and d_{ij} represent the elements of C and D respectively,

$$d_{ij} = 2p(b_i c_{jk} b_k - c_{ik} b_k b_j) + pc_{ij} + q\delta_{ij}.$$

The equation $\lambda = Dx$ implies $q = (x\lambda)/(xx)$, and $p = \pm\sqrt{(xx)(\lambda\lambda) - (x\lambda)^2}/(xx)$. Further, if $\sigma_i = \lambda_i - qx_i - pc_{ik}x_k$, it implies $\sigma_i c_{jk}\sigma_k - \sigma_j c_{ik}\sigma_k = 4p^2[b_i c_{jk} b_k - c_{ik} b_k b_j][(c_{hk} b_k x_h)^2 + (bx)^2]$, and $(\sigma\sigma) = 4p^2[(c_{hk} b_k x_h)^2 + (bx)^2]$. Then unless $(\sigma\sigma) = 0$, two solutions have been found:

$$d_{ij} = 2p(\sigma_i c_{jk}\sigma_k - \sigma_j c_{ik}\sigma_k)/(\sigma\sigma) + pc_{ij} + q\delta_{ij},$$

where the σ's, q, and the ambiguous[2] p are the stated functions of x and λ.

If $(\sigma\sigma) = 0$, then $\lambda_i = qx_i + pc_{ij}x_j$. Such vectors may be called exceptional, and each fails to define a single pair of matrices.

In R_4, the matrix $D = qI + pBCB$ reduces to

$$\begin{pmatrix} q & -\beta & -\gamma & -\delta \\ \beta & q & e_1 e_2 \delta & -e_1 e_2 \gamma \\ \gamma & -e_1 e_2 \delta & q & e_1 e_2 \beta \\ \delta & e_1 e_2 \gamma & -e_1 e_2 \beta & q \end{pmatrix},$$

where $\beta = 2e_1 p(\frac{1}{2} - b_1^2 - b_2^2)$, $\gamma = 2e_1 p(-b_2 b_3 + e_1 e_2 b_1 b_4)$, and $\delta = 2e_1 p(-b_2 b_4 - e_1 e_2 b_1 b_3)$. This matrix is left or right handed according as e_1 and e_2 have the same or opposite sign; hence in this case the choice of sign of p is immaterial, and even an exceptional vector defines a unique matrix D.

In any R_n the exceptional vectors have the properties:
(i) If any tangent vector to a geodesic is exceptional,

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1) T. Muir and W. H. Metzler: Theory of Determinants, p. 585.

2) If $q \neq 0$, then choosing the sign of p so that $pq \geq 0$ makes the solution unique and such that $d_{ij}(-\lambda, x) = -d_{ij}(\lambda, x)$. However, if q=0, there seems to be no satisfactory criterion for choosing the sign of p so as to obtain this last property.

then all its tangent vectors are exceptional.

(ii) The locus of points at which vectors of a given field $\lambda=Dx$ are exceptional is two hyperplanes of dimension n-2 and 2.

The first of these can be shown for any geodesic (3) by assuming its tangent vector at s=0 to be exceptional, say $\lambda_o = qIa+pCa$, and then showing that every tangent vector satisfies $\lambda = qIx+pCx$.

To prove the second, let the field be $\lambda=(q_1I+p_1BCB)x$. At a point x where an exceptional vector is defined, $\lambda = (q_2I+p_2C)x$. It follows that $q_1=q_2$, $p_1=\pm p_2$, so that if $p_1\neq 0$ then $(BCB\pm C)x=0$.

The hyperplane (BCB-C)x=0 has equations $(b_ic_{jk}b_k-c_{ik}b_kb_j)x_j=0$. These are equivalent to the system $c_{jk}b_kx_j=0$, $b_jx_j=0$, which defines a hyperplane of dimension n-2. This hyperplane insects (BCB+C)x=0 only in the point x=0, so the dimension of the latter must be 2 or 0. (It cannot be 1 since its matrix is skew-symmetric.) Computation of respective elements shows BCBC=CBCB, so (BCB+C)(BCB+C)C=-2(BCB+C). If BCB+C were non-singular, it would follow that BCB+C=2C. But then $b_ic_{jk}b_k-c_{ik}b_kb_j=0$, so that b=0, which is impossible. Hence BCB+C is singular, and (BCB+C)x=0 is of dimension 2.

Property (ii) enables one to define a "parallel" transport independent of the path, where vectors are "parallel" if they belong to the same field $\lambda=Dx$, provided only that the path avoid the exceptional hyperplanes of that field. In view of the dimensionality of these hyperplanes, they can always be avoided.

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VITA

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